MA 425/525 final review problems
Version as of December 3rd.

The final will be on Monday, December 9th, as scheduled on `https://www.purdue.edu/registrar/pdf/exam/current_final_exam_schedule.pdf`. No notes, books, or electronic devices will be allowed. Most of the exam will be closely based on problems, or on parts of problems, from the list below. Justify your answers. Please let me know if you have a question or find a mistake.

1. Find integers a and b such that $(1 + i)^5 = a + bi$. Same question for $(1 - i)^{10}$, $(-1 + i)^{15}$, $(-1 - i)^{20}$.

2. How many solutions does $z^{2/3} = \sqrt{3} + i$ have? Find and plot them.

3. Let $f(z) = z^5 + 20z^2 + 22$. For each positive integer n, determine how many zeroes of f lie in the set $A_n = \{z \in \mathbb{C} \colon n - 1 < |z| < n\}$.

4. Let p be a real number and let $f(z) = (e^z + e^{-z})^{-1}$. What is the radius of convergence of the Taylor series of f centered at p? Same question for $f(z) = (z^2 - 2z + 5)^{-1}$ and $f(z) = (z^2 - 2z + 5)^{-1}(z^2 + 4z + 5)^{-1}$.

5. Find the number of solutions of $z^{100} = e^z$ in $\{z \colon \operatorname{Re} z < 0\}$.

6. Let a be a nonzero real number. Let Γ_1 be the segment from -1 to ai and let Γ_2 be the segment from ai to 1. Let $f(z) = \displaystyle\sum_{n=-1}^{1} z^n$. Evaluate $\displaystyle\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$.

7. Let m and n be positive integers, and let $f(z) = (m^2 + 4)z^n + e^z$. How many zeroes does f have in $\{z \colon |z| < 1\}$ and what are their orders?

8. Let H be the right half plane $\{z \colon -\pi/2 < \operatorname{Arg} z < \pi/2\}$. Find a one-to-one analytic function f from H onto H such that $f(3 + i) = 9 + 2i$.

9. Let S be the sector $\{z \colon -3\pi/4 < \operatorname{Arg} z < -\pi/2\}$. Find a one-to-one analytic function f from S onto S such that $f(-1 - 2i) = -1 - 3i$.

10. Evaluate
$$\int_C f(z)dz,$$
where $f(z) = \tan((1 + i)z)$, and C is the circle $|z| = 2$, oriented clockwise.

Hints and some answers:

1. Use $(1+i)^2 = 2i$ to simplify the first one, and similar equations for the others. You can look up answers to these and similar problems here `https://www.wolframalpha.com/input?i=%28-1-i%29%5E20`.

2. Use polar form. The answer is two solutions and they are $2 + 2i$ and $-2 - 2i$.

3. See Example 4 on page 178. You can check your answer here `https://samuelj.li/complex-function-plot` `#z%5E5%2B20*z%5E2%2B22`.

4. Use Theorem 1 on page 123. Given p, how big can R be such that f is analytic on the disk $\{z \colon |z - p| < R\}$? For the first one, the answer is $\sqrt{p^2 + \frac{\pi^2}{4}}$.

5. See Example 3 on page 176. If this is giving you trouble, try replacing 100 by a smaller number and checking your answer here `https://samuelj.li/complex-function-plotter/` `#z%5E3-exp(z)`.

6. Choose a branch of log according to whether a is positive or negative, similar to problems 1 and 2 on the last homework.

7. Use the definition on page 128 to determine how many zeroes can have order ≥ 2, without trying to find the actual zeroes; this means solving $f(p) = f'(p) = 0$. To find the number of zeroes, use Rouché's theorem. For exmaple, if $m = 3$ and $n = 4$, then the answer is four simple zeroes.

8. $f(z) = 3z - i$.

9. $f(z) = \sqrt[4]{4z^4 + 56}$, where the branch of $\sqrt[4]{\ }$ is taken such that $\{z \colon -\pi < \operatorname{Arg} z < 0\}$ is mapped to S.

10. $2\pi(1 + i)$.